Competing Inheritance Paths in Dependent Type Theory

a case study in functional analysis

Reynald Affeldt, Cyril Cohen, Marie Kerjean,
Assia Mahboubi, Damien Rouhling, Kazuhiko Sakaguchi

AIST / Inria / U. Tsukuba

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What this Presentation is About

inheritance in hierarchies of structures

- Hierarchies of structures are important to organize formalizations
- They are implemented with type classes or unification hints
- Problems for developers: performance/predictability,
  not all issues are properly documented
- In this talk, we discuss problems caused by competing definitions
- In particular, we identify **forgetful inheritance** as a solution to
  the problem of competing inheritance paths for poorer structures

Motivation for this Work

analysis with the Coq proof assistant

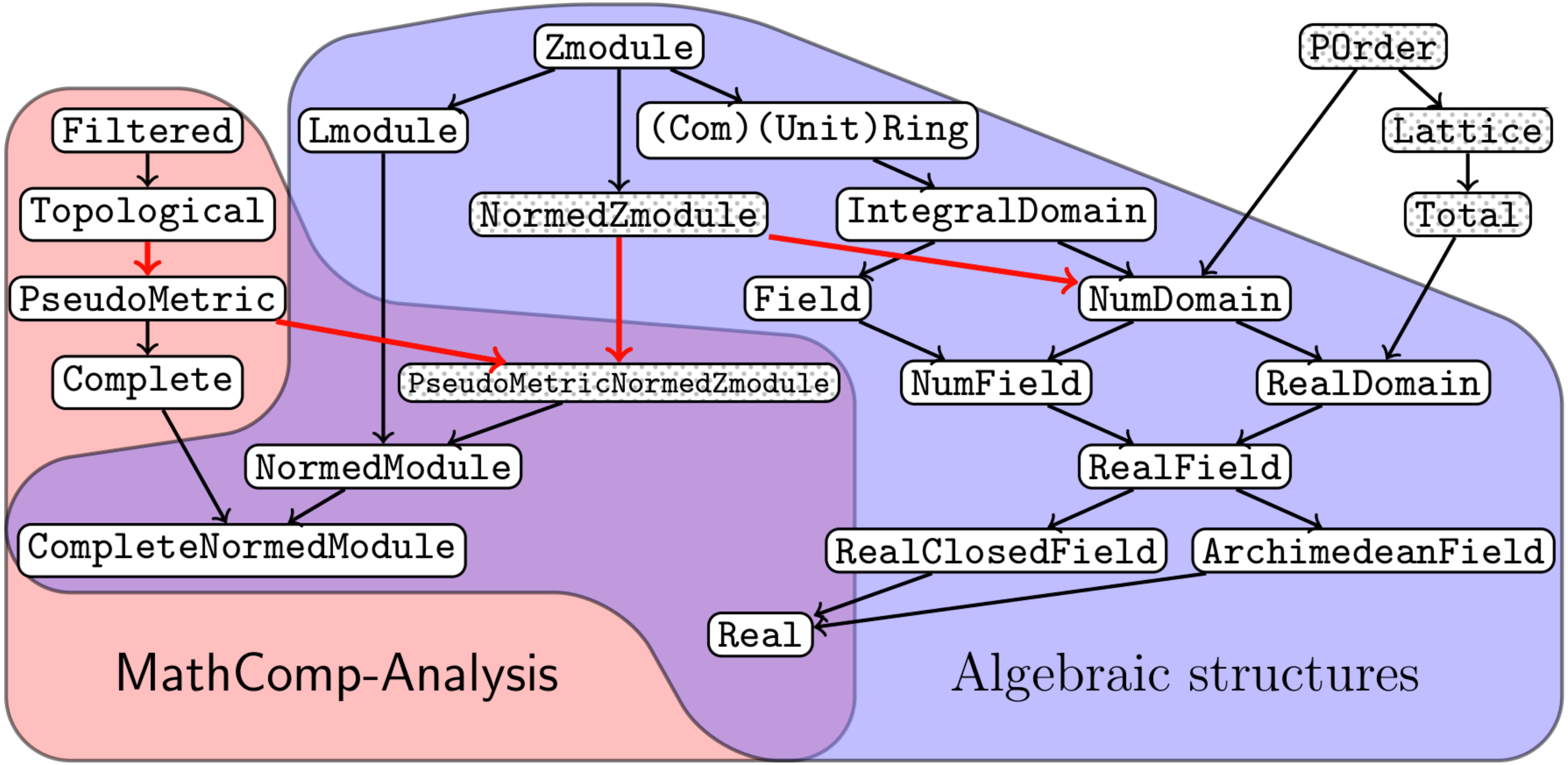
- MathComp is the library for algebra used to formalize the odd order theorem by Gonthier et al. [ITP 2013]

- Coquelicot is a library for real analysis by Boldo et al. [MCS 2015]
  - It extends and improves the standard library of Coq

- MathComp-Analysis is a work-in-progress that extends MathComp for functional analysis.
  - It is inspired by Coquelicot but comes with original features [JFR 2018]

MathComp-Analysis Hierarchy

the concrete result of this presentation (teaser)



Competing Inheritance Paths in Dependent Type Theory

outline

1. Background: Hierarchies of Structures
2. Problem: Extend MathComp with Analysis
3. Forgetful Inheritance for Competing Inheritance Paths
4. Application to MathComp-Analysis
5. Conclusion

$\boxed{\text{module}}$

Mathematical Structure

using a packed class [Garillot et al., 2009]

1️⃣ Record **isModule** T := IsModule {**minus_op** : T -> T -> T}. } Mixin = operator + properties

2️⃣ Structure **module** := Module {

module_carrier : Type ;

module_isModule : isModule module_carrier }.

} Structure = carrier + mixin

3️⃣ Definition **minus** (M : module)

: module_carrier M -> module_carrier M -> module_carrier M

:= minus_op _ (module_isModule M).

} operator lifted from the mixin to the structure

4️⃣ Notation “x - y” := (minus _ x y). } Notation with a hole ≈ overloading

Structure Inference
with unification hints

module □−□ $\xrightarrow{\text{canonical}}$ Z_module

1️⃣ Definition Z_module := Module Z (IsModule Z Z.sub). } structure instance

2️⃣ Canonical Z_module. } the instance is canonical

Check forall x y : Z, x - y = x - y. $\xrightarrow[\text{Notations}]{\text{Unset Printing}}$ minus ? x y

Z

module

module_carrier ?

type equation

? ≡ Z_module $\xleftarrow{\text{solution}}$ module_carrier ? ≡ Z

Inheritance of Structures

using the notion of class

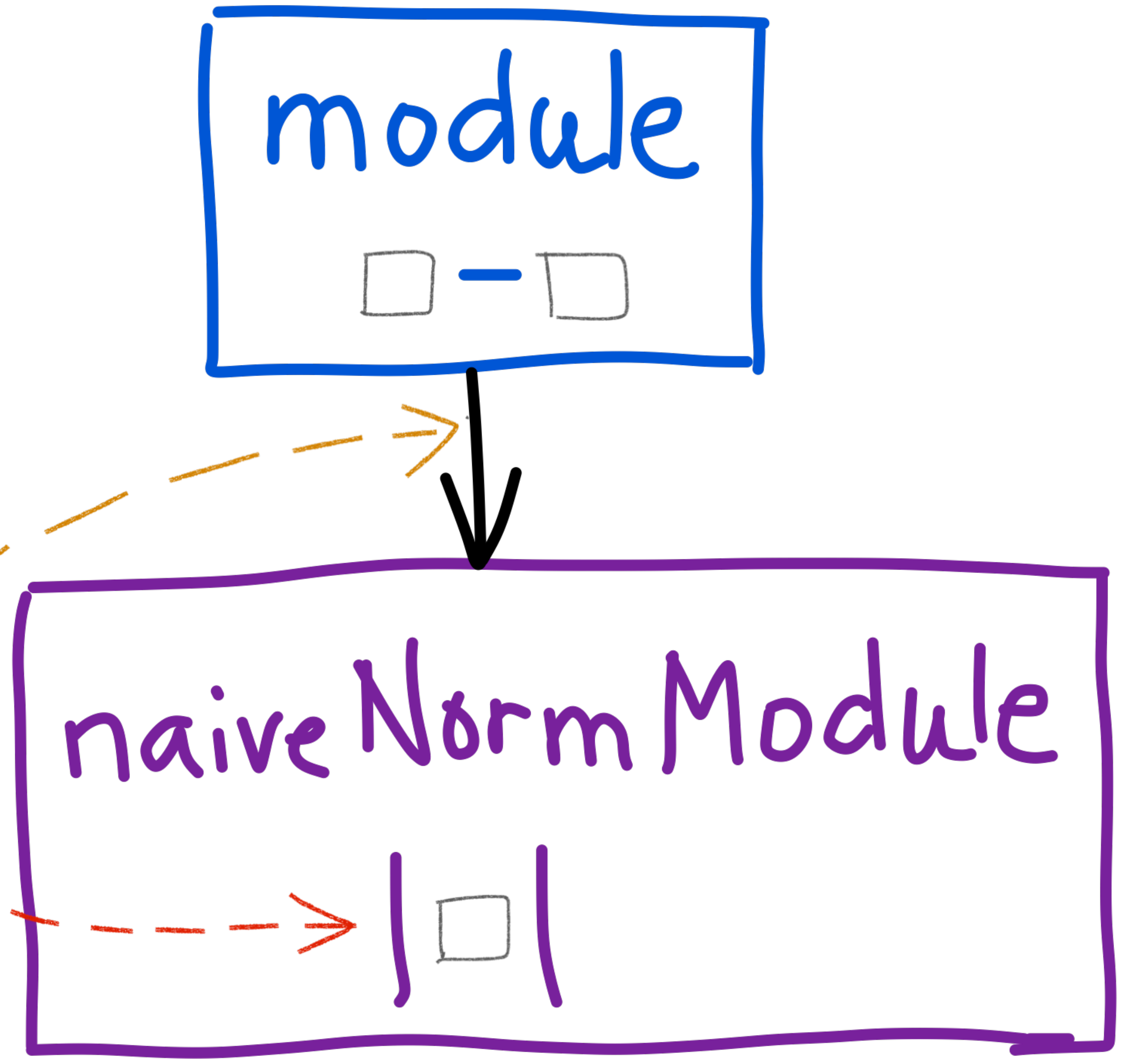
1️⃣ Record **naiveNormMixin** (T : module) := NaiveNormMixin { mixin

naive_norm_op : T -> nat }.

Record **isNaiveNormModule** (T : Type) := IsNaiveNormModule {

nbase : isModule T ; class ≈ inheritance structure

nmix : naiveNormMixin (Module _ nbase) }.



2️⃣ Structure **naiveNormModule** := NaiveNormModule { structure

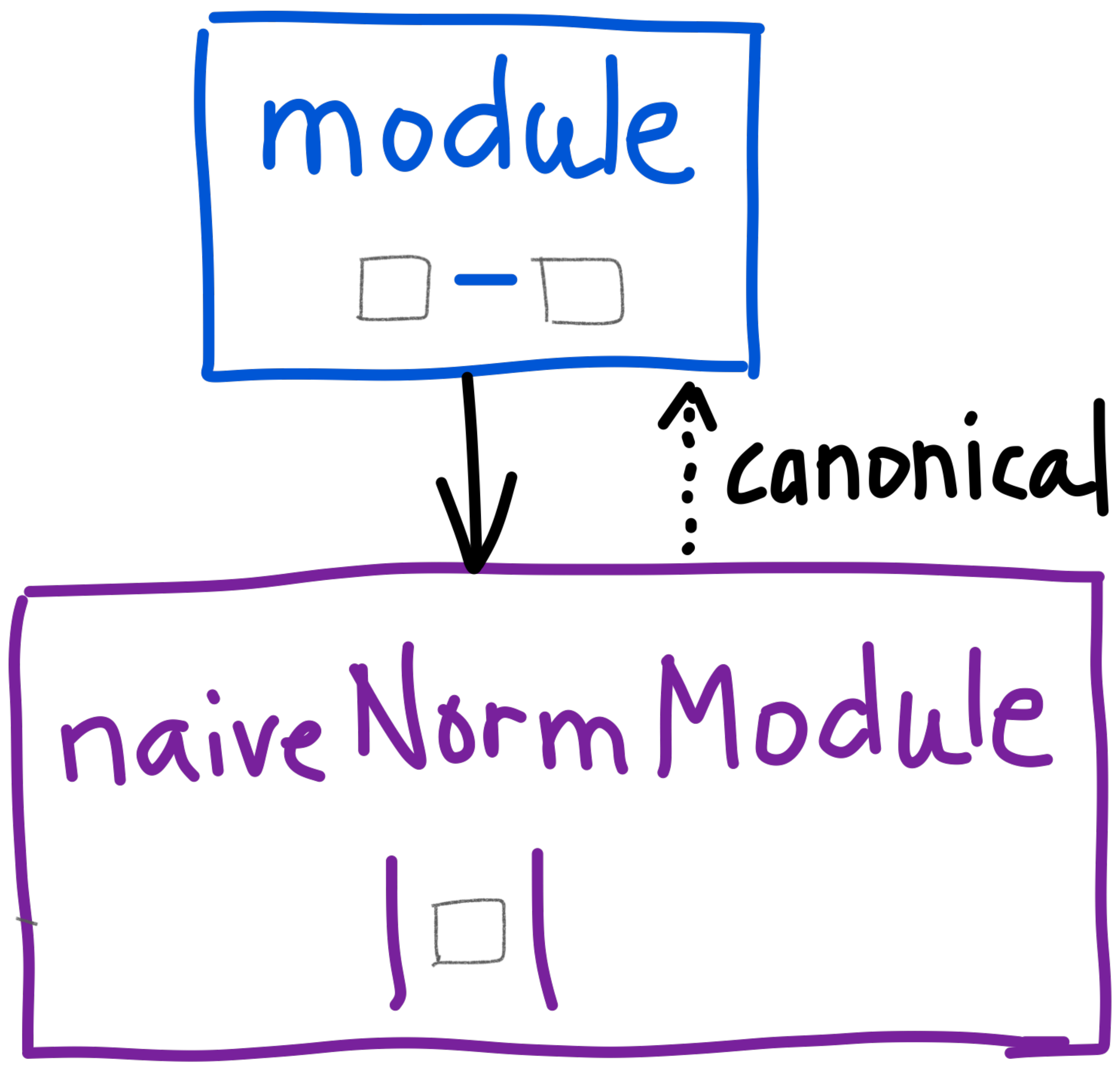
naive_norm_carrier :> Type ;

implicit coercion

naive_normModule_isNormModule : isNaiveNormModule naive_norm_carrier }.

Inference in Presence of Inheritance

using unification hints



Check forall (N : naiveNormModule) (x y : N),

module_carrier ?

x - y = x - y.

implicit coercion

naive_norm_carrier N

KO

Canonical **naiveNorm_isModule** (N : naiveNormModule) :=

Module N (nbase _ (naive_normModule_isNormModule N)).

module_carrier ? ≡ naive_norm_carrier N

OK

naiveNorm_isModule N

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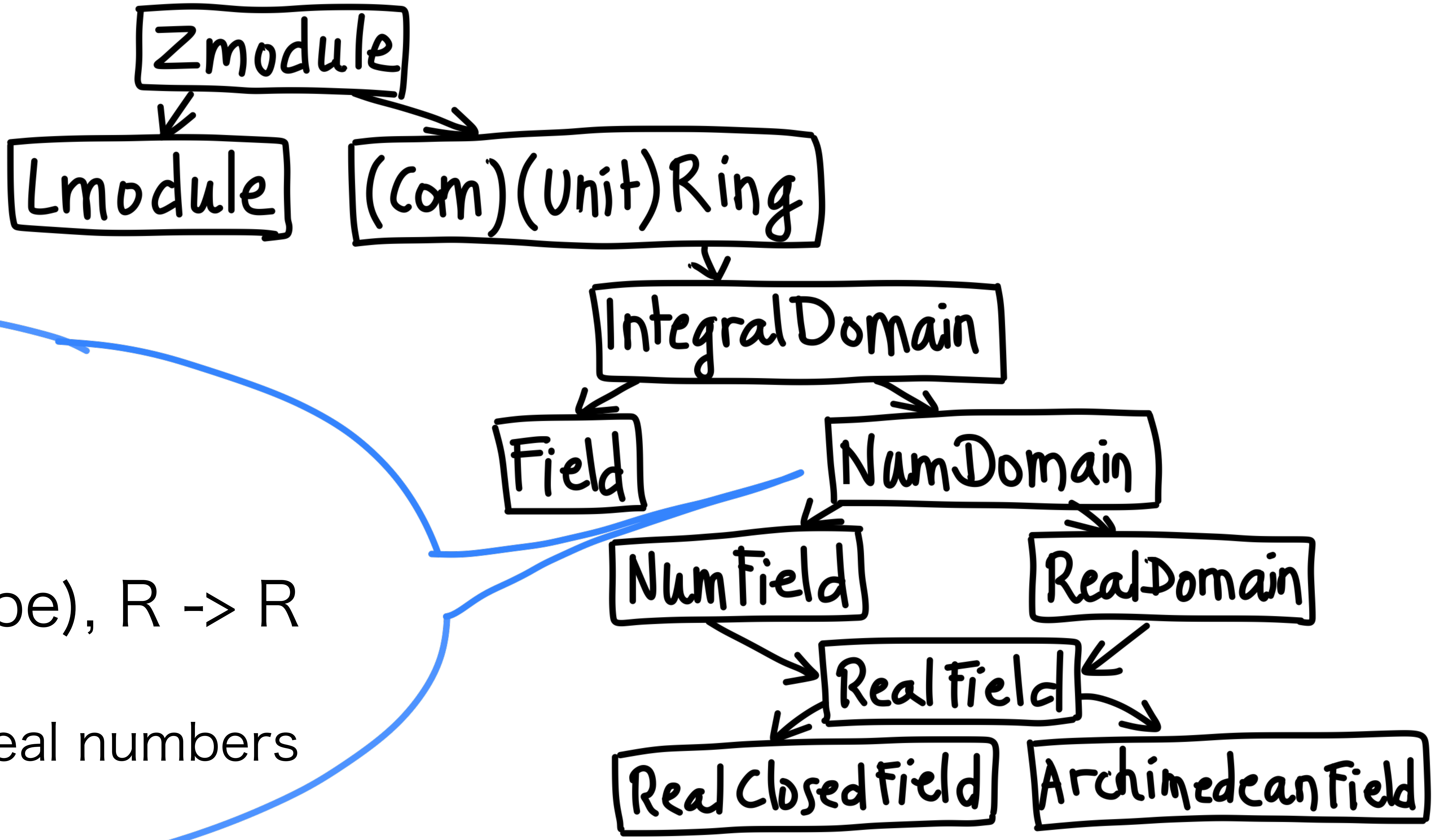
The MathComp Library
(excerpt, before this work)



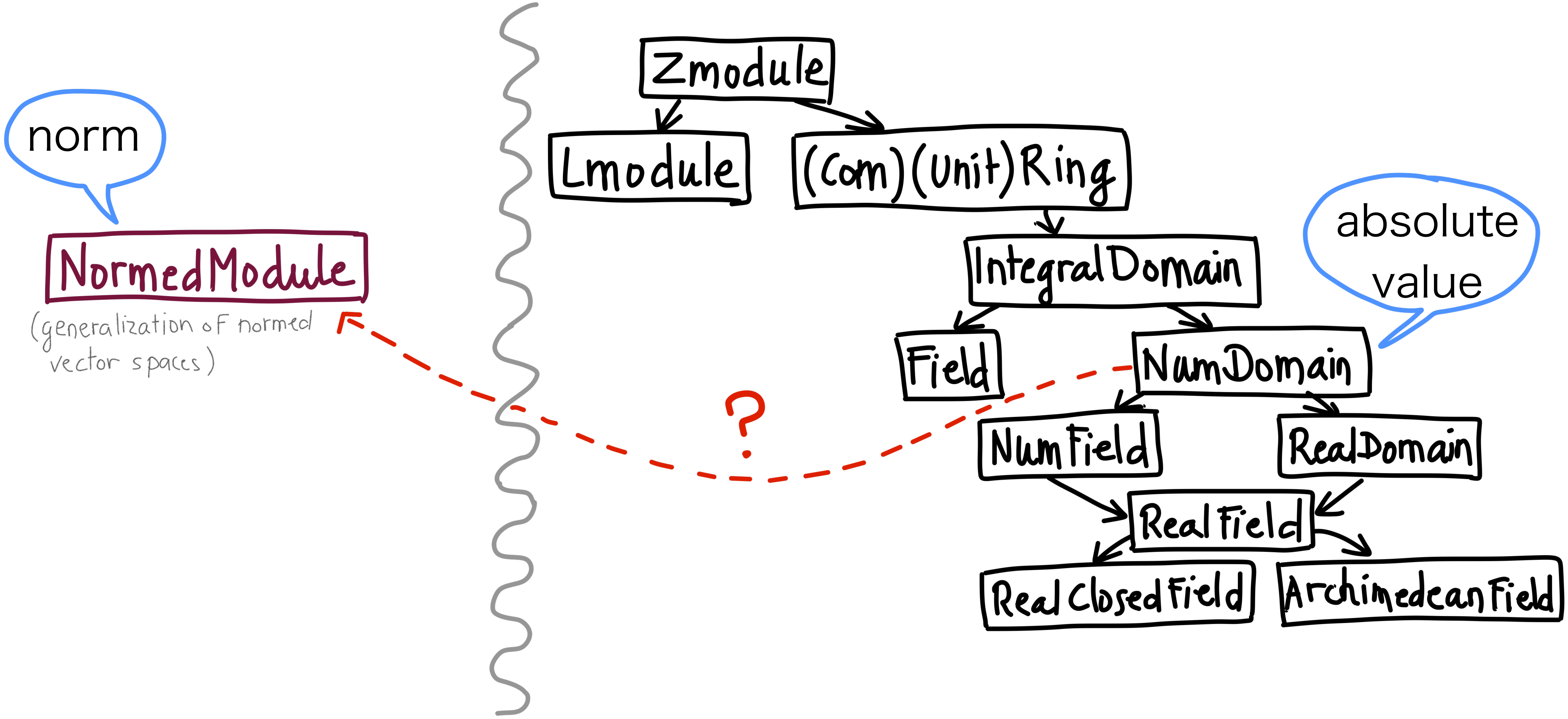
absolute value of type:

forall (R : numDomainType), R -> R

Sample instances: integers, real numbers

Towards Analysis with MathComp

issue #1: properties of differentiability, continuity



Towards Analysis with MathComp

issue #2: topology

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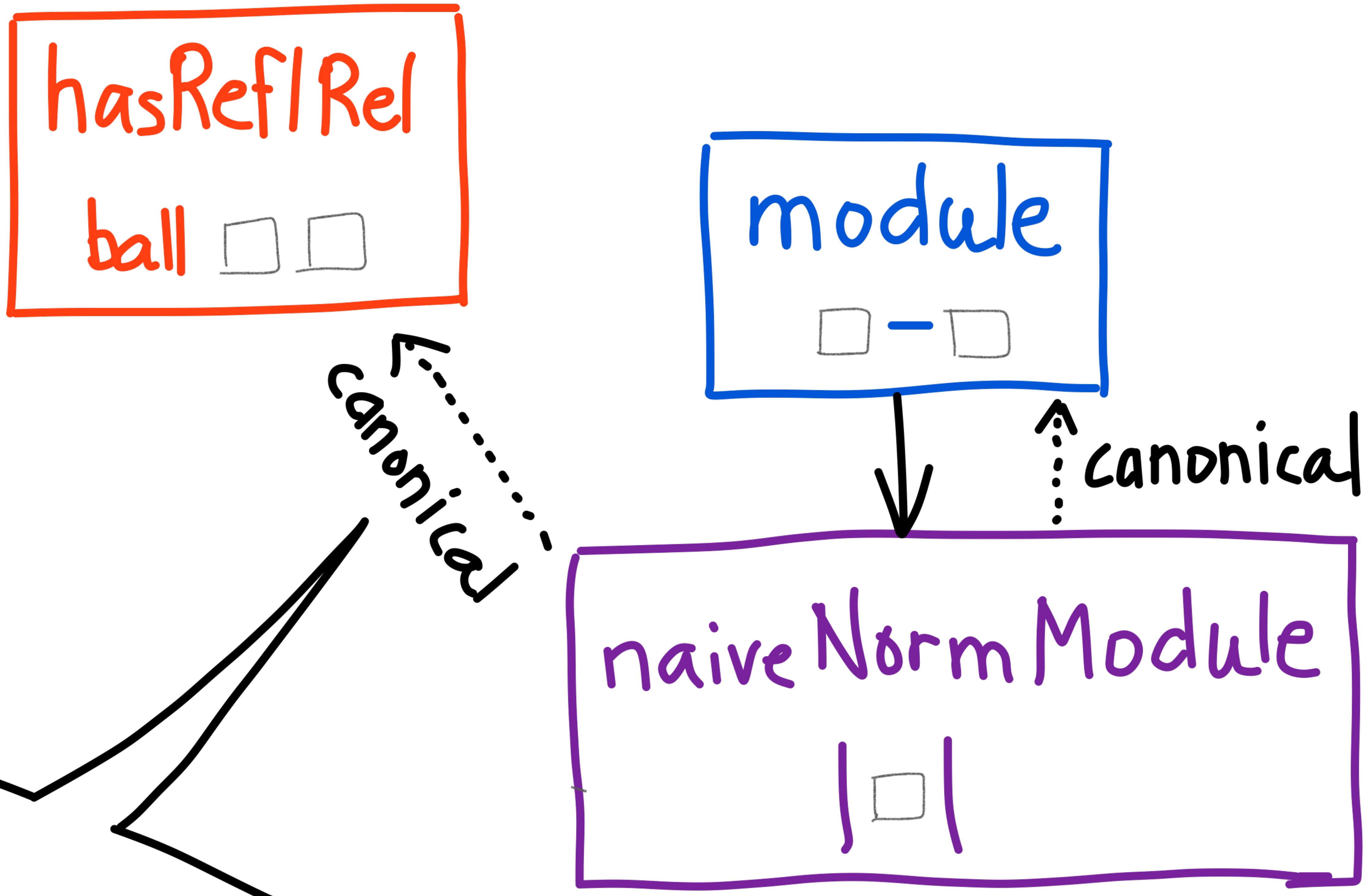
Hierarchy Extension

a new structure with a notion of ball



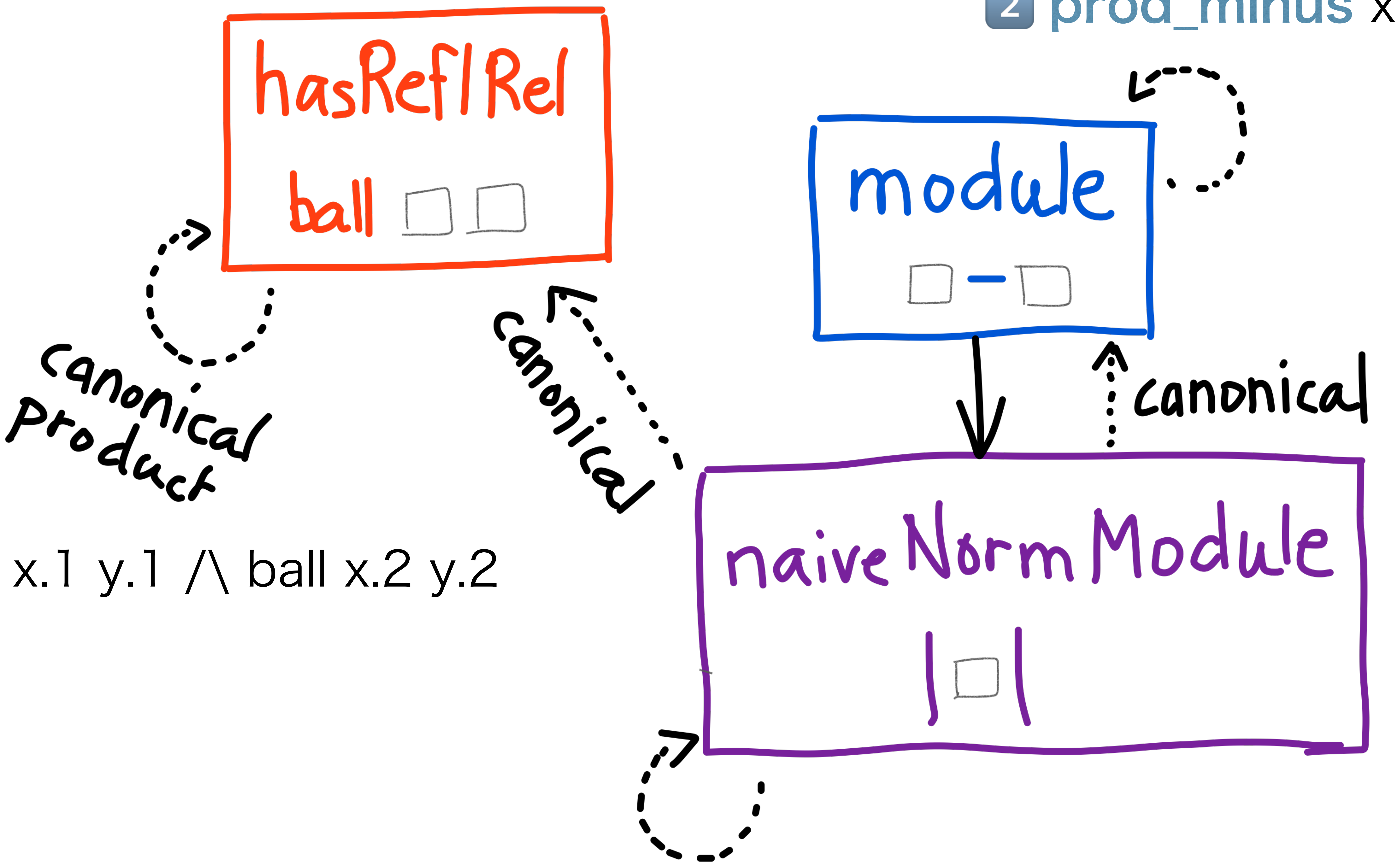
Notion of ball defined using norm

```
Variable N : naiveNormModule.

Definition norm_ball (x : N) := fun y : N => |x - y| ≤ 1.

Definition naiveNormModule_isReflRel := IsReflRel N norm_ball … .

Canonical nnorm_hasReflRel := HasReflRel N naiveNormModule_isReflRel
```

Hierarchy Extension

canonical products for all structures

2️⃣ prod_minus x y := (x.1 - y.1, x.2 - y.2).



1️⃣ prod_ball x y := ball x.1 y.1 /\ ball x.2 y.2

3️⃣ prod_nnorm x := max |x.1| |x.2|

A Puzzling Inference Failure

Example failure

Hypothesis → (Pball : forall V : naiveNormModule, forall v : V, **P** (ball v))

(W : naiveNormModule) (w : W * W)

Goal → : **P** (ball w).

Proof. Fail apply Pball. Abort.

Shouldn't this succeed since there is a canonical way to build products? 🤔

A Puzzling Inference Failure

prod_naiveNormModule W W

Example failure   nnorm_hasReflRel ?

(Pball : forall V : naiveNormModule, forall v : V, **P** (ball v))

(W : naiveNormModule) (w : W * W)

: **P** (ball w).

Proof. Fail apply Pball. Abort.

prod_hasReflRel (nnorm_hasReflRel W)
(nnorm_hasReflRel W)

A Puzzling Inference Failure

Example failure

(Pball : forall V : naiveNormModule, forall v : V, **P** (ball v))

(W : naiveNormModule) (w : W * W)

: **P** (ball w).

Proof. Fail apply Pball. Abort.

prod_naiveNormModule W W

nnorm_hasReflRel ?

$\text{ball\_op}\ x\ y \triangleq |x-y| \leq 1$

$\not\equiv$ KO

not definitionally equal

$\text{ball\_op}\ x\ y \triangleq \text{ball}\ x_1\ y_1 \wedge \text{ball}\ x_2\ y_2$

prod_hasReflRel (nnorm_hasReflRel W) (nnorm_hasReflRel W)

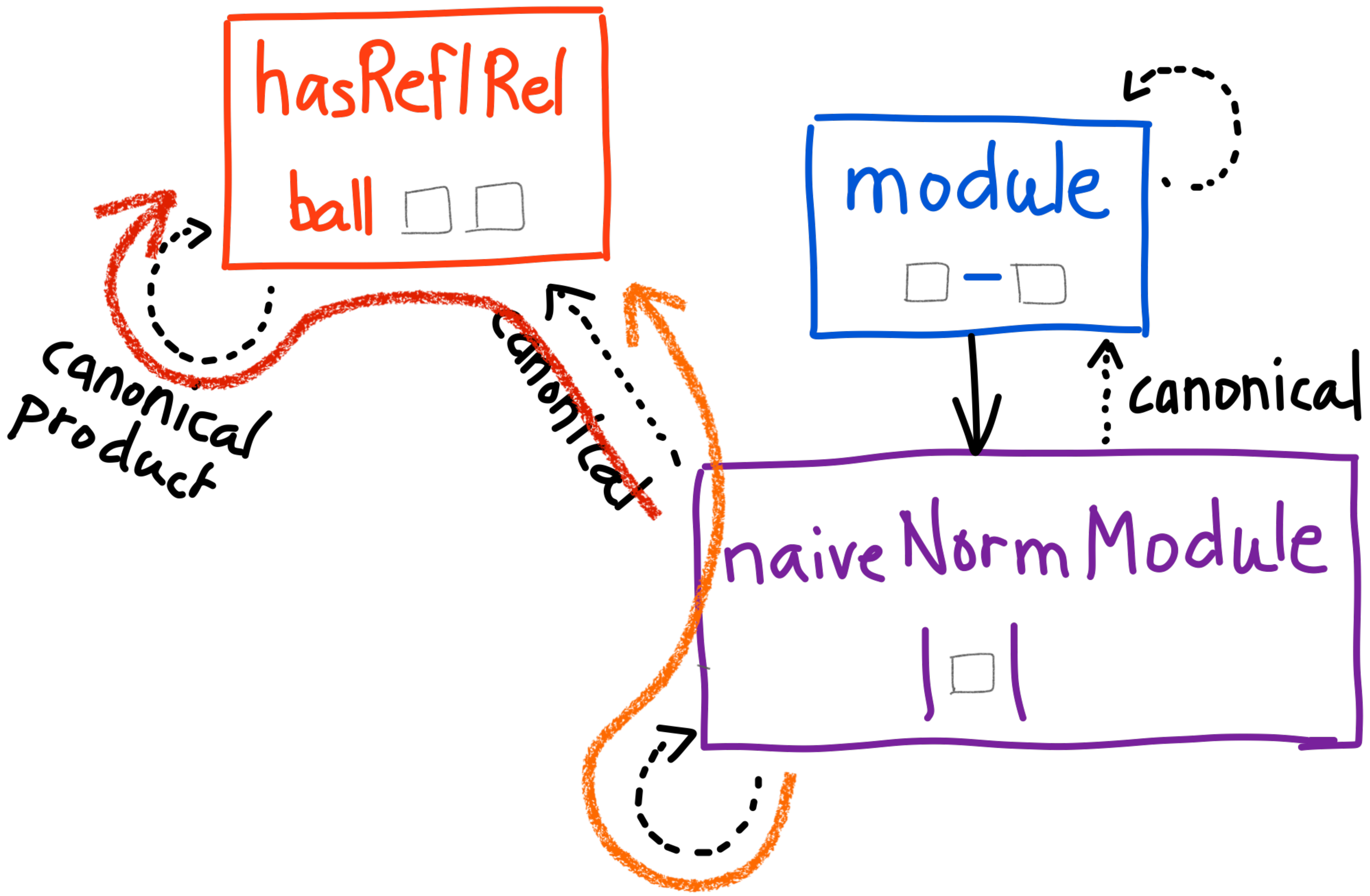
Problem of Competing Inheritance Paths

diagrammatically

(hypothesis in the example)

$$\begin{tikzcd}
\texttt{naiveNormModule} * \texttt{naiveNormModule} \arrow[r, "\texttt{prod_naiveNormModule}"] \arrow[d, "\texttt{nnorm_hasReflRel}"'] \arrow[dr, phantom, "\not\circlearrowleft"] & \texttt{naiveNormModule} \arrow[d, "\texttt{nnorm_hasReflRel}"] \\
\texttt{hasReflRel} * \texttt{hasReflRel} \arrow[r, "\texttt{prod_hasReflRel}"'] & \texttt{hasReflRel}
\end{tikzcd}$$

(goal in the example)



Forgetful Inheritance

how to fix the hierarchy

has Refl Rel as a parameter

1 Record **normMixin** (T : module) (m : isReflRel) := NormMixin {

norm_op : T -> nat ;

compatibility norm ⇕ ball

norm_ball_opP : forall x y, ball_op _ m x y <-> norm_op (x - y) ≤ 1 }.

2 Record **isNormModule** (T : Type) := IsNormModule {

base : isModule T ;

bmix : isReflRel T ;

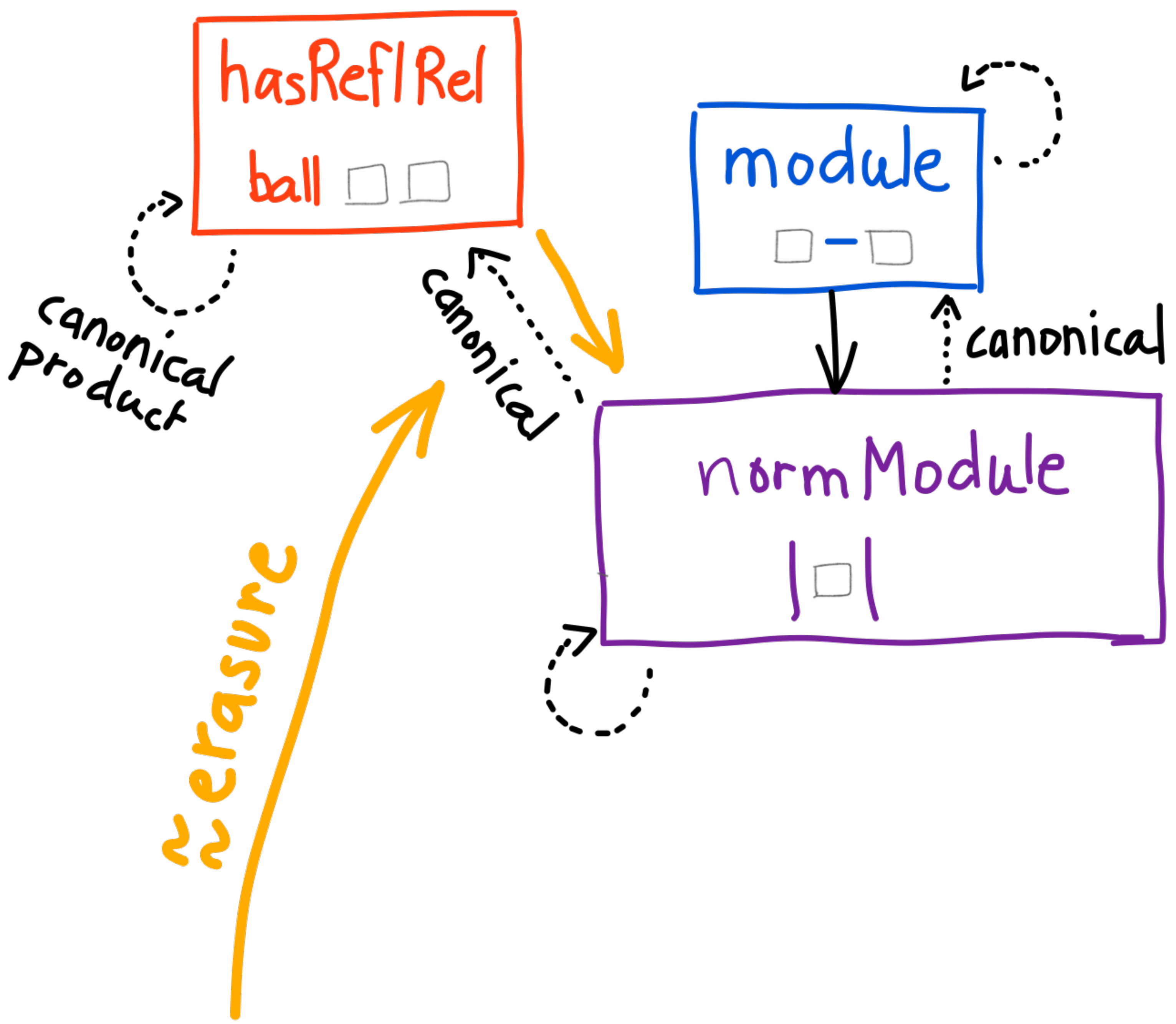
has Refl Rel mixin included in the class

mix : normMixin (Module _ base) bmix }.

3 Structure **normModule** := NormModule {

norm_carrier :> Type ;

normModule_isNormModule : isNormModule norm_carrier }.



4 Canonical **norm_hasReflRel** (N : normModule) :=

HasReflRel N (bmix _ (normModule_isNormModule N))

Competing Inheritance Paths with Type Classes

same problem, same solution

Type classes in Lean

Type classes in Coq

Packed classes in Coq

```
-------------------------------
-- Section 2.2               --
-- isModule is in fact has_sub --
-------------------------------
----------------
-- Section 2.3 --
----------------
universe u
```

```
From Coq Require Import ssreflect ZArith.
Reserved Notation "| x |"   (at level 30).
Reserved Notation  "x ~~ y" (at level 30).
```

```
From Coq Require Import ssreflect ZArith.
Reserved Notation "| x |"   (at level 30).
Reserved Notation  "x ~~ y" (at level 30).

(**************)
(* Section 2.1 *)
(**************)
Record flatNormSpace := FlatNormSpace {
  carrier : Type ;
  fminus : carrier -> carrier -> carrier;
  fnorm : carrier -> nat;
  fnormP : forall x : carrier, fnorm (fminus x x) = 0}.

Check fminus.
(* fminus : forall f : flatNormSpace,                    *)
(*   carrier f -> carrier f -> carrier f        *)
Check fnormP.
(* fnormP : forall (f : flatNormSpace) (x : carrier f), *)
(*   fnorm f (fminus f x x) = 0                        *)

Lemma Z_normP (n : Z) : Z.abs_nat (Z.sub n n) = 0.
Proof. by rewrite Z.sub_diag. Qed.
Definition Z_flatNormSpace := FlatNormSpace Z Z.sub Z.abs_nat Z_normP.

(**************)
(* Section 2.2 *)
(**************)
Record isModule T := IsModule {minus_op : T -> T -> T}.
Structure module := Module {
  module_carrier : Type ;
  module_isModule : isModule module_carrier}.

Definition minus {M : module} := minus_op _ (module_isModule M).
Notation "x - y" := (minus x y).

Definition Z_isModule : isModule Z := IsModule Z Z.sub.
Definition Z_module := Module Z Z_isModule.

Fail Check forall x y : Z, x - y = x - y.

Canonical Z_module.
Check forall x y : Z, x - y = x - y.
Print Canonical Projections.
(* the line Z <- module_carrier (Z_module) tries to solve *)
(* the equation `module_carrier ?M =?= Z`                *)

(* begin omitted from the paper *)
Coercion module_carrier : module >-> Sortclass.
(* end omitted from the paper   *)
```

https://math-comp.github.io/competing-inheritance-paths-in-dependent-type-theory/

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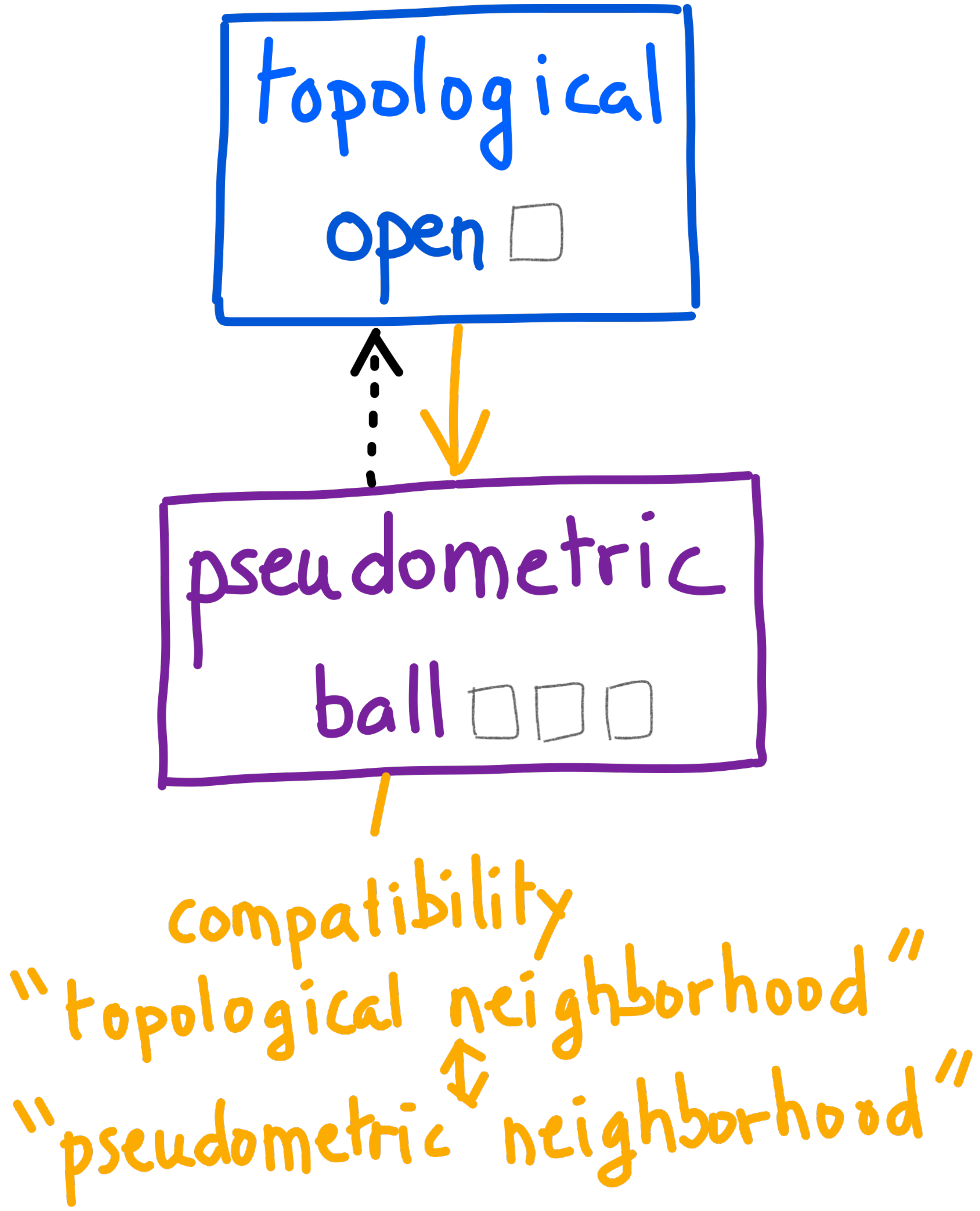
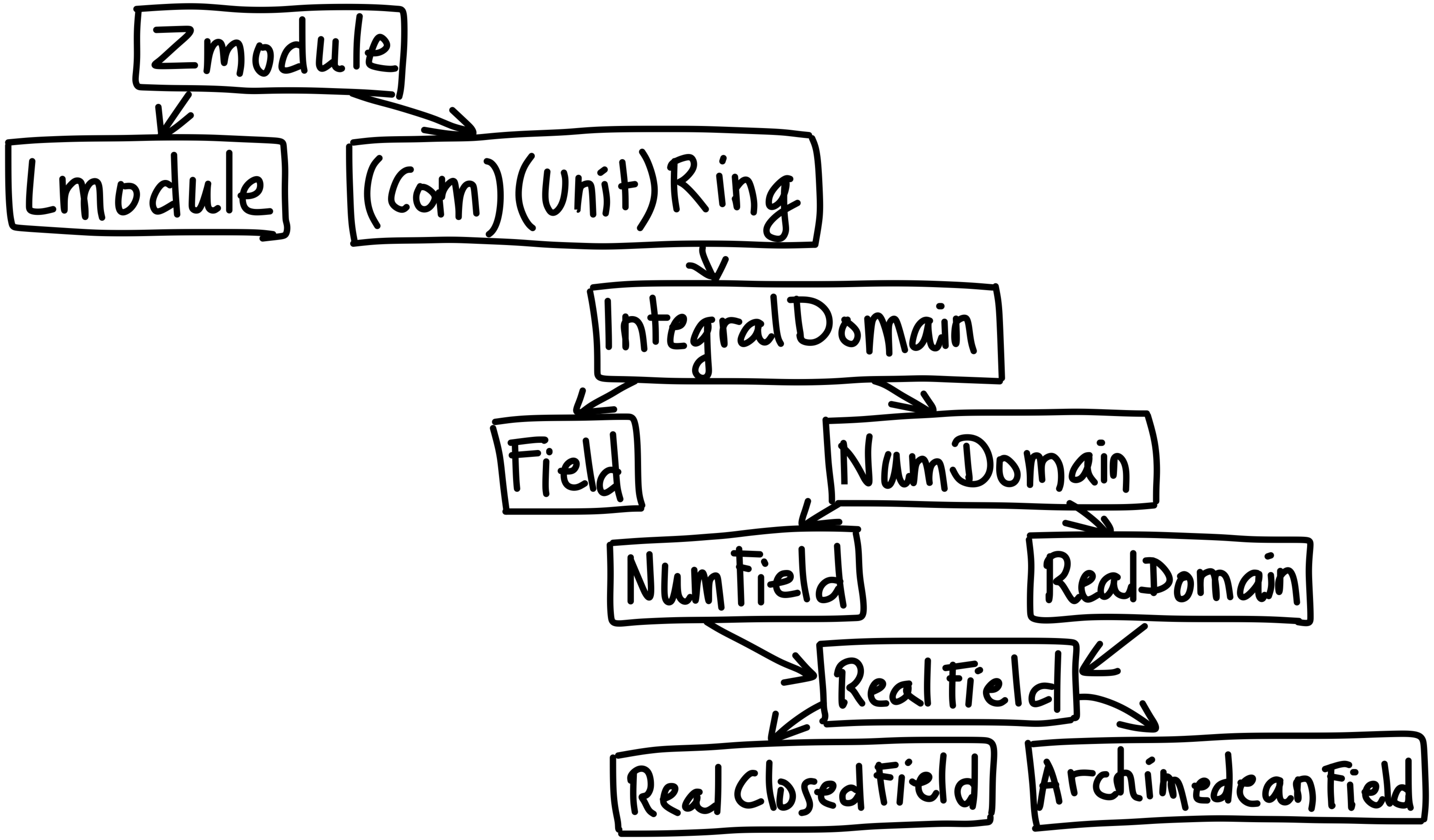
Forgetful Inheritance

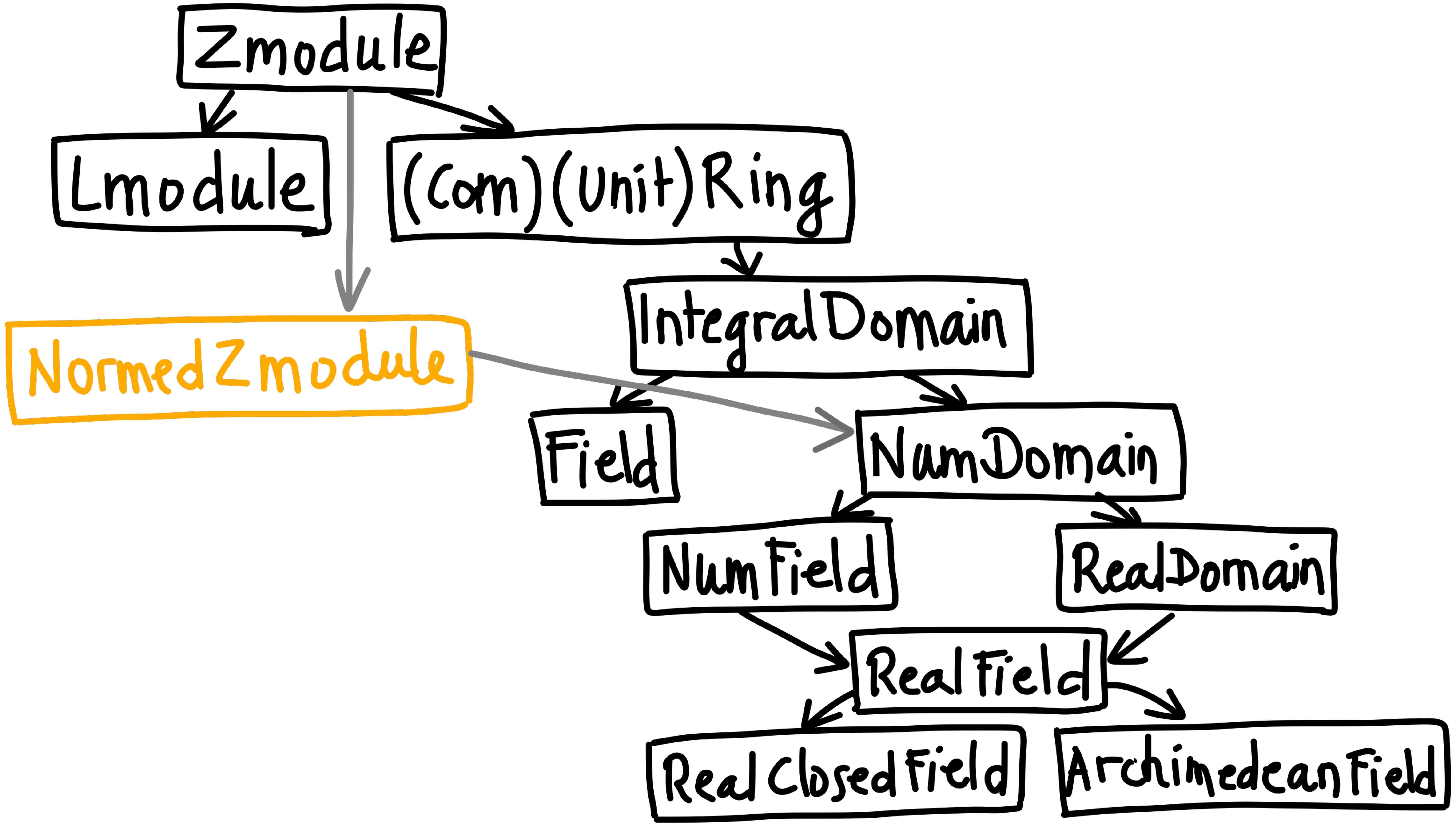
from pseudometric spaces to topological spaces





Unify Absolute Value and Norm

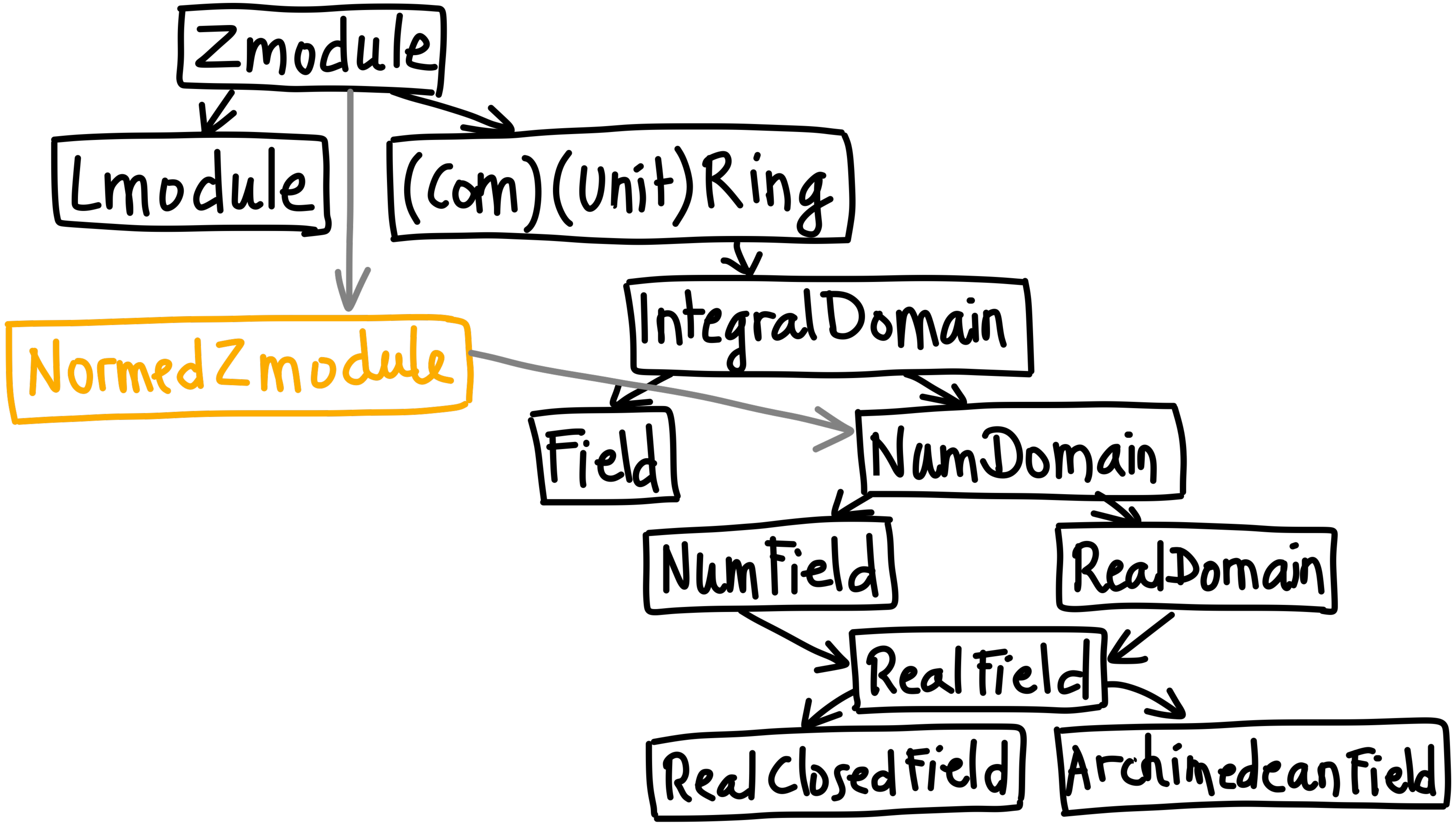
step #1: factorize with normed Abelian groups



Unify Absolute Value and Norm

step #2: solve mutual dependency problem

properties of the norm require the codomain of the norm to be a normed Abelian group and an order (e.g., triangle inequality)... 🤔

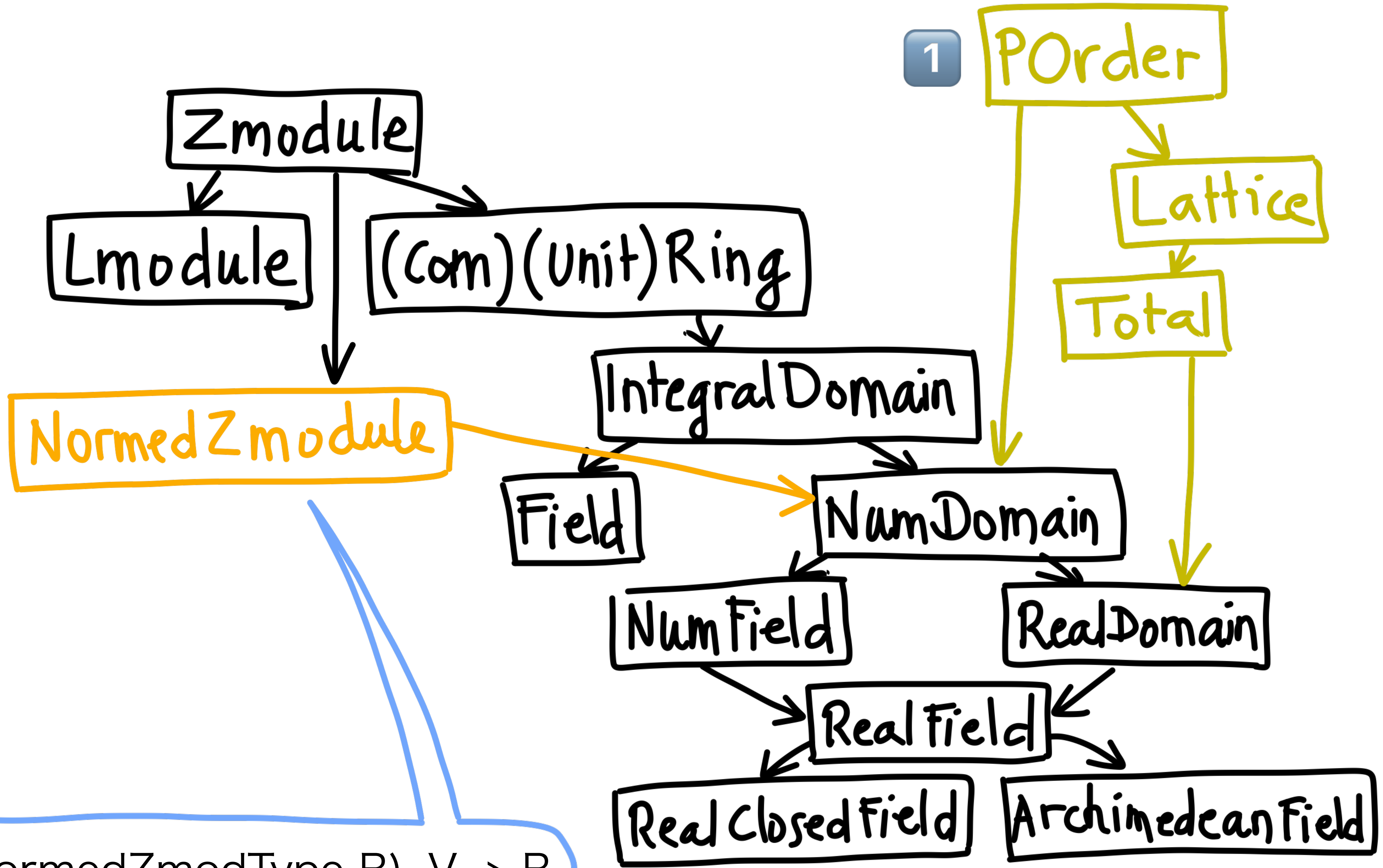


Forgetful Inheritance
from numerical domain to normed Abelian group

2️⃣ the class of NumDomain changed to include the mixin of NormedZmodule

3️⃣ Forgetful inheritance:

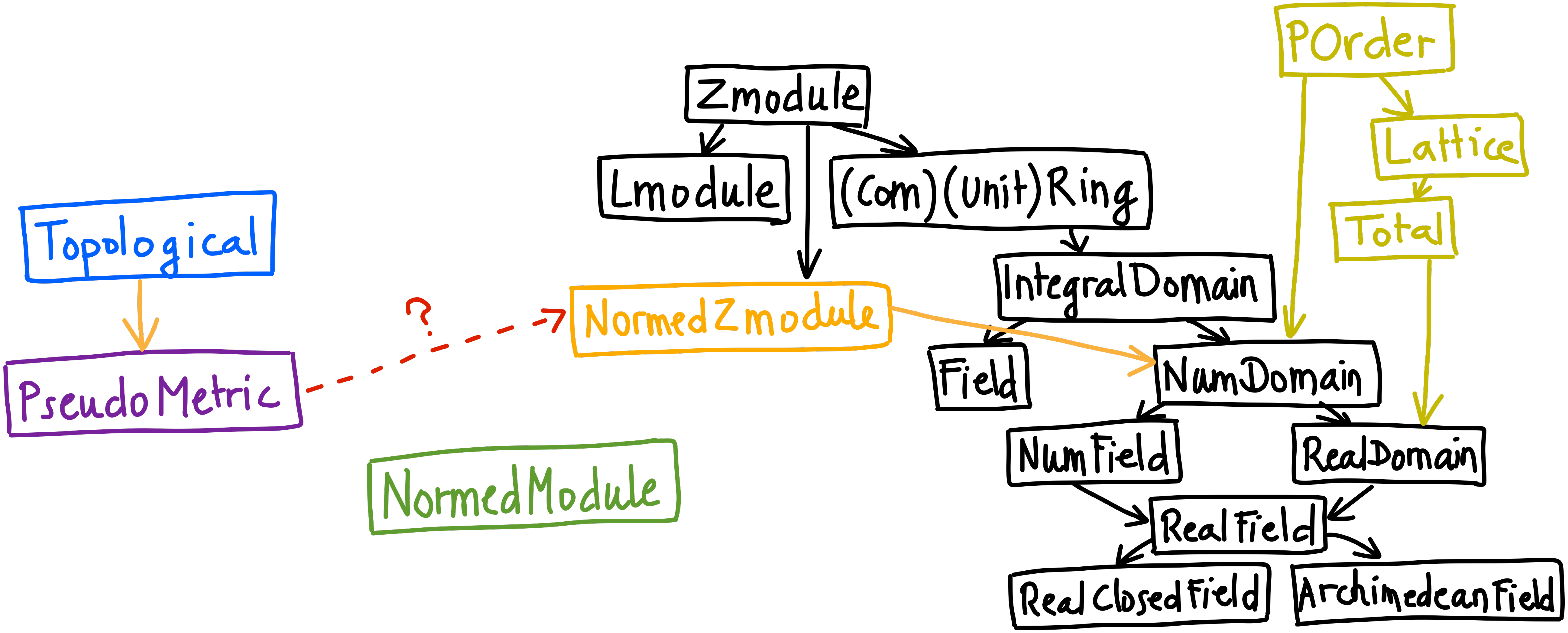
1. the class of NormedZmodule is parameterized by NumDomain and features the mixin of the norm
2. inheritance from NormedZmodule to NumDomain is declared



norm : forall (R : numDomainType) (V : normedZmodType R), V -> R

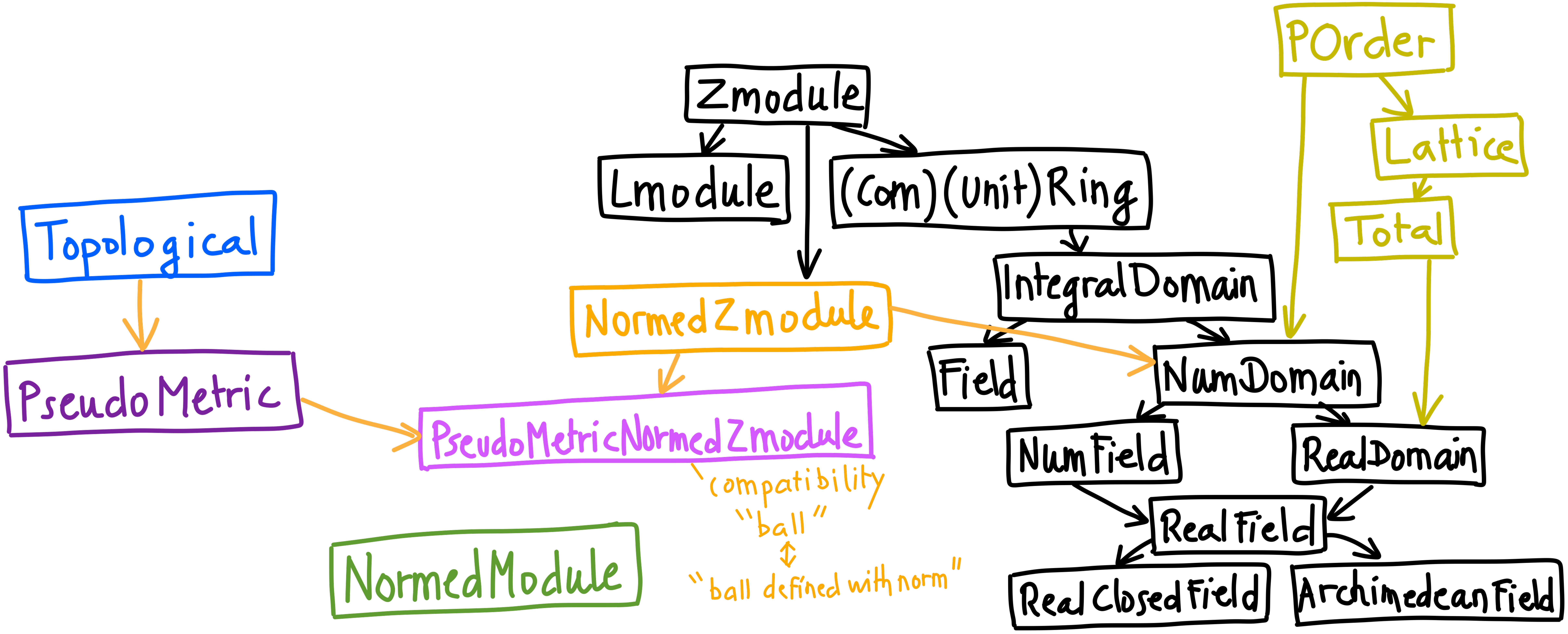
How do we Integrate Normed Modules?

final step



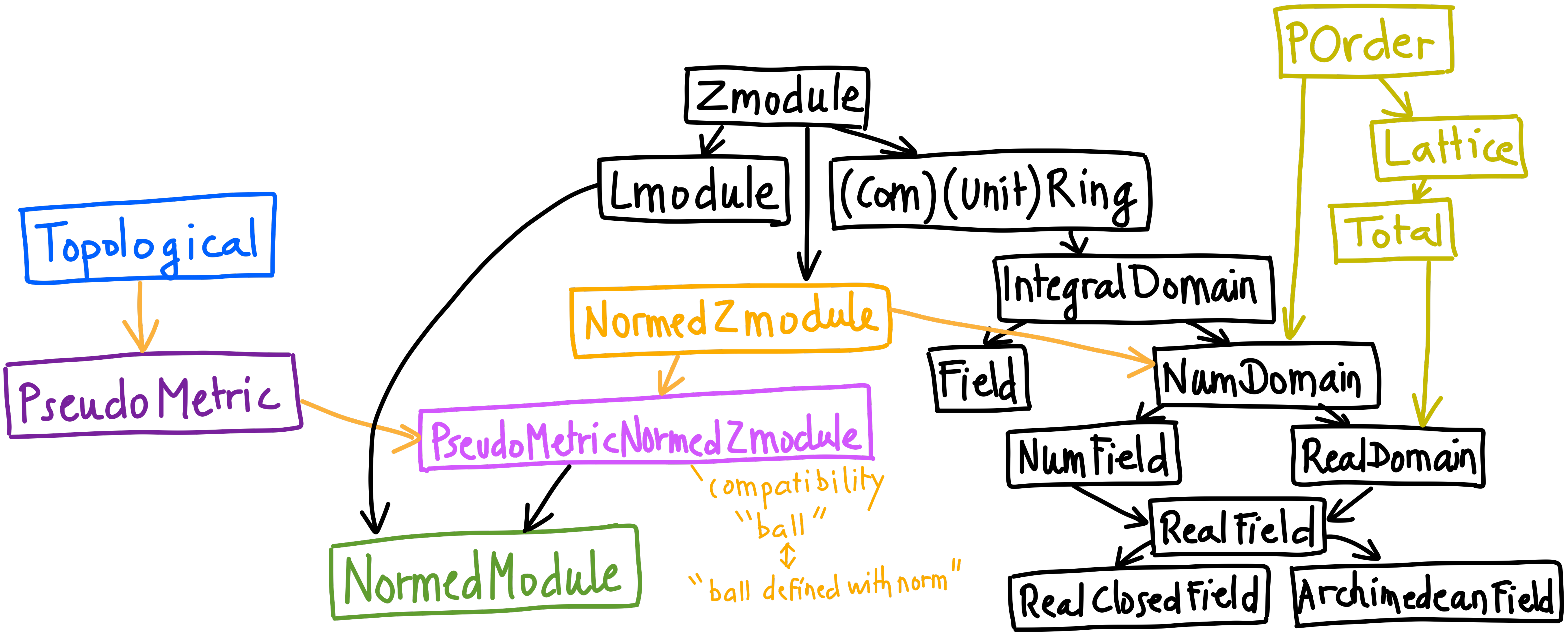
Forgetful Inheritance

from normed modules to pseudometric spaces



Forgetful Inheritance

from normed modules to pseudometric spaces



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Related Work

- HOL Light and Isabelle/HOL have more real and complex analysis
  - different foundations: we use dependent types here
- Forgetful inheritance was already mentioned
  - original description of packed classes [TPHOLs 2009] alludes to it
  - Buzzard et al. [CPP 2020] discuss a similar issue, on a specific example
- Packed classes + forgetful inheritance is verbose
  - Sakaguchi [IJCAR 2020] developed an automated checking tool ← on Friday
  - Cohen et al. [FSCD 2020] are working on automated generation ← tomorrow

Conclusion

summary of main contributions

1️⃣ forgetful inheritance using packed classes

(several examples, comparison with type classes)

2️⃣ the MathComp-Analysis hierarchy

(enhancement of MathComp,

applications of forgetful inheritance)

Available: theories of Bachmann-Landau notation, of differentiability